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Ray's theorem revisited: a fixed point free firmly nonexpansive mapping in Hilbert spaces

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Abstract
We give another proof of a strong version of Ray's theorem ensuring that every unbounded closed convex subset of a Hilbert space admits a fixed point free firmly nonexpansive mapping.

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1 Ray's theorem and its strong version

In 1965, Browder [1] showed the following fixed point theorem for nonexpansive mappings in Hilbert spaces.

Theorem 1.1 (Browder's theorem [1]) *Let C be a nonempty closed convex subset of a Hilbert space H. If C is bounded, then every nonexpansive self-mapping on C has a fixed point.*

Ray [2] showed that the converse of Browder's theorem holds.

Theorem 1.2 (Ray's theorem [2]) *Let C be a nonempty closed convex subset of a Hilbert space H. If every nonexpansive self-mapping on C has a fixed point, then C is bounded.*

Later, Sine [3] gave a simple proof of Theorem 1.2 by applying a version of the uniform boundedness principle and the convex combination of a sequence of metric projections onto closed and convex sets.

Recently, Aoyama *et al.* [4], obtained a counterpart of Theorem 1.2 for λ-hybrid mappings in Hilbert spaces by using the following strong version of Ray's theorem.

Theorem 1.3 (A strong version of Ray's theorem [4]) *Let C be a nonempty closed convex subset of a Hilbert space H. If every firmly nonexpansive self-mapping on C has a fixed point, then C is bounded.*

It should be noted that Theorem 1.3 was actually shown by using Theorem 1.2 in [4]. See also [5, 6] on generalizations of Theorem 1.3 for firmly nonexpansive type mappings in Banach spaces.

In this paper, motivated by the papers mentioned above, we give another proof of Theorem 1.3 by using a version of the uniform boundedness principle and a single metric projection onto a closed and convex set. Since every firmly nonexpansive mapping is nonexpansive, Theorem 1.3 immediately implies Theorem 1.2.

2 A fixed point free firmly nonexpansive mapping

Throughout this paper, every linear space is real. The inner product and the induced norm of a Hilbert space H are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The dual space of a Banach space X is denoted by X^*. The following is a version of the uniform boundedness principle.

Theorem 2.1 (see, for instance, [7]) *If C is a nonempty subset of a Banach space X such that $x^*(C)$ is bounded for each $x^* \in X^*$, then C is bounded.*

Let C be a nonempty closed convex subset of a Hilbert space H. Then a self-mapping T on C is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; firmly nonexpansive [8, 9] if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$ for all $x, y \in C$. The set of all fixed points of T is denoted by $\mathrm{F}(T)$. The mapping T is said to be fixed point free if $\mathrm{F}(T)$ is empty. It is well known that for each $x \in H$, there exists a unique $z_x \in C$ such that $\|z_x - x\| \leq \|y - x\|$ for all $y \in C$. The metric projection P_C of H onto C, which is defined by $P_C x = z_x$ for all $x \in H$, is a firmly nonexpansive mapping of H onto C. This fact directly follows from the fact that the equivalence

$$z = P_C x \iff \sup_{y \in C} \langle y - z, x - z \rangle \leq 0 \tag{2.1}$$

holds for all $(x, z) \in H \times C$. See [10–12] for more details on nonexpansive mappings.

We first show the following lemma.

Lemma 2.2 *Let C be a nonempty closed convex subset of a Hilbert space H, a be an element of H, and T be the mapping defined by $Tx = P_C(x + a)$ for all $x \in C$. Then T is a firmly nonexpansive self-mapping on C such that*

$$\mathrm{F}(T) = \left\{ u \in C : \langle u, a \rangle = \sup_{y \in C} \langle y, a \rangle \right\}. \tag{2.2}$$

Proof Since P_C is firmly nonexpansive, we have

$$\|Tx - Ty\|^2 \leq \big\langle P_C(x + a) - P_C(y + a), (x + a) - (y + a) \big\rangle = \langle Tx - Ty, x - y \rangle$$

for all $x, y \in C$. Thus T is a firmly nonexpansive self-mapping on C. Fix any $u \in C$. According to (2.1), we know that

$$Tu = u \iff \sup_{y \in C} \big\langle y - u, (u + a) - u \big\rangle \leq 0 \iff \langle u, a \rangle = \sup_{y \in C} \langle y, a \rangle$$

and hence (2.2) holds. □

Using Theorem 2.1 and Lemma 2.2, we give another proof of Theorem 1.3.

Proof of Theorem 1.3 If C is unbounded, then Theorem 2.1 implies that $x^*(C)$ is unbounded for some $x^* \in H^*$. Since H is a real Hilbert space, we have $a \in H$ such that $\sup_{y \in C} \langle y, a \rangle = \infty$. By Lemma 2.2 and the choice of a, the mapping T defined as in Lemma 2.2 is a fixed point free firmly nonexpansive self-mapping on C. □

Competing interests
The author declares that he has no competing interests.

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